Voronoi Growth Model Applied to Green Island Formation in Wildland Fires

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Abstract

Wildland fires in relatively homogeneous terrains and vegetation motivates this work which considers the formation of green islands, regions of vegetation that become enclosed and eventually engulfed by the fire. A Voronoi growth model adequately represents the spreading of a fire from isolated point-like sources. Using this model, it is possible to predict the green island formations. First, we introduce a condition for when three simultaneous point-like ignition sources can form a simple green island and then present an algorithm to infer when a complex green island is formed by more than three sources at a time t. Finally, we comment on the extension of Voronoi growth model to other fire source shapes.

Keywords: Voronoi growth model, wildland fire, green island formation, landscape ecology.

1 Introduction

The occurrence of wildland fire had a significant growth in recent years. In particular it has been much observed in Brazil, where wildland fires and *queimadas*[1] are frequent and also during summer in the US, Australia and in European countries. Not only do wildland fires [2] have implications to the global environment, but they also incur a large financial cost.

In the literature of wildland fires several authors have been concerned with fire spread models, giving most of their attention to the fire process itself [18, 9, 10, 13] and only more recently some have given closer attention to the study of the spatial patterns and geometrical aspects [6, 21, 3, 14].

Modelling the landscape geometry of burned out areas due to surface fire on vegetation is quite important as it can provide a better understanding of the fire behavior [2] and its dynamics. This is essential for fire control and suppression, identification of *refuge zones* and landscape investigation of spatial patterns disturbance.

In the papers [17, 16] it has been shown that, starting with very simple assumptions, a Voronoi diagram structure [12, 15] naturally emerges as the landscape geometry generated by the spread out of wildland fire. In this article, assuming the Voronoi growth model, we wish to focus our attention to the specific problem of *green island* (refuge zones) formation, which correspond to isolated regions of vegetation that will ultimately be burned out if the fire is not suppressed.

Initially, a simple situation will be considered where the fuel bed is composed of a vegetation with a homogeneous distribution over a regular topography. In the absence of wind and other severe weather conditions and for a set of simultaneous or synchronous fire ignition sources, it is shown that the geometrical construction of Voronoi diagram is a natural structure to model the landscape geometry of burned out vegetation areas. Under these circumstances the fire wavefront would simply propagate with radial (or nearly circular) symmetry. The intersection of the fire wavefronts of two neighboring ignition sources would occur at the bisector of the segment that joins the two ignition points. This corresponds to the very basic principle of the Voronoi diagram geometrical structure. Latter we consider a situation where the fire ignition sources are no longer simultaneously lit and are randomly spatially distributed. This situation is likely to occur by the process called "spotting" [2].

In Section 2 an overview of the geometric construction of Voronoi diagrams are given, in Section 3 the simple green island definition and formation are provided, in Section 4 the complex green island definition and its formation are treated and, finally, Section 5 is devoted to comments and conclusions.

[1] Man made fires, either prescribed or not.

2 Voronoi diagrams

In this section we provide a brief introduction to the planar Voronoi diagram geometrical construction.

2.1 Ordinary Voronoi diagram

The planar ordinary Voronoi diagram (OVD) [12] is defined as a division of the plane into regions according to the principle of the *nearest neighbor*. Let us consider the Euclidean distance $d(p, p_i)$ from a point p to a set of non-collinear points p_i in the plane. A *Voronoi region* $R(p_i) \equiv R_i$ generated by a point p_i is defined as $R_i = \{p\,; d(p, p_i) \le d(p, p_j), \forall p_i \neq p_j\}$. The Voronoi diagram $V(P)$ for a set of points $P = \{p_1, p_2, \ldots, p_n\}$, is defined as the union of all Voronoi regions $V(P) = \bigcup_{i=1}^{n} R(p_i)$. The points p_i are called *Voronoi generators.* The edges between two adjacent Voronoi regions are called *Voronoi edges* and the points where 3 or more Voronoi edges meet are called *Voronoi vertices*. We say that a Voronoi generator p_i is adjacent to p_j when their Voronoi regions share a common edge. According to its definition, the Voronoi diagram is such that any point on the edge of two neighboring regions is equidistant from the corresponding Voronoi generators.

The *region of dominance* of p_i over p_j is defined as:

$$D(p_i, p_j) = \{\boldsymbol{x};\, \|\boldsymbol{x} - \boldsymbol{x_i}\| \le \|\boldsymbol{x} - \boldsymbol{x_j}\|,\, i \neq j\}, \tag{1}$$

where $\boldsymbol{x}$, $\boldsymbol{x_i}$, and $\boldsymbol{x_j}$ are the position vectors of the points p, p_i and p_j, respectively. The regions of dominance for a set of generators are equivalent to the Voronoi diagram for these generators. Therefore, a planar ordinary Voronoi diagram may be constructed by *halfplanes* taking into account that the Voronoi region $R(p_i)$ can be constructed equivalently by:

$$R(p_i) = \bigcap_{j \in I_n \setminus \{i\}} D(p_i, p_j),\; I_n = 1, 2, \ldots, n. \tag{2}$$

One important property of the OVD is that the Voronoi vertex is the center of the circle that passes exactly through at least three Voronoi generators that create the vertex. Figure 1 shows an OVD for 20 generators.

2.2 Voronoi diagrams from growth models

Growth models are processes that produce spatial patterns as a consequence of the growth of a set of points P on the plane or in higher dimensions. Suppose we have a set of points or sites $P = \{p_1, p_2, \ldots, p_n\}$ on the plane. Let us assume that:

1. Each point p_i is located simultaneously.

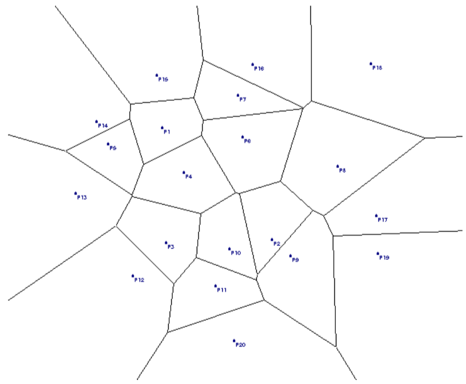

Figure 1: Planar OVD for 20 generators.

2. Each point p_i is fixed at its original position throughout the growth process.

3. The growth starts simultaneously and with the same growth rate v in all directions from p_i.

4. The growth process ceases in one direction whenever the region of growing from point p_i comes into contact with the growing region of a point p_j, $i \neq j$.

This set of assumptions gives rise to a Voronoi diagram structure known as *Voronoi Growth Model* [12] or *cell model*, as the intersection of growth points occurs at the bisector of them, which corresponds to the basic principle of the ordinary Voronoi diagram stated in the previous subsection. This kind of modelling has been applied in some fields like physics [8], geology [22], geography [11] and also in ecology [7].

The growth model assumptions above can be modified to describe more general situations where the sites are no longer simultaneously distributed and/or the growth rate is different for each site. This will lead to more complex Voronoi growth models. In these cases the Voronoi diagrams generated are no longer the OVD, they become generalized Voronoi diagrams for circular objects [19] (see Figure 2), as will be seen in the next section. Also, in [19] it was considered the case where the sites could be inside another growth region. Note that this situation cannot happen in the fire growth model because it is assumed that a spark cannot take place inside an area that has already been burned out.

We will see later that under certain simple assumptions the fire spread process can be regarded as a Voronoi Growth Model or an extension of it.

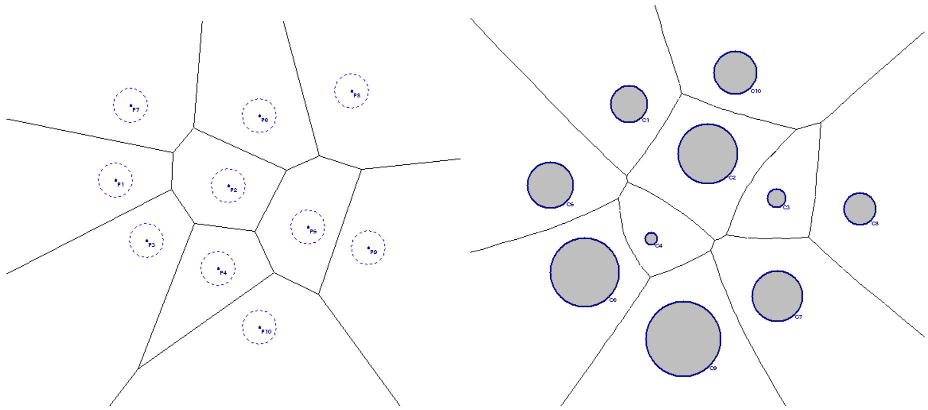

Figure 2: Voronoi growth model diagram for synchronous (left) and asynchronous (right) ignition sources.

2.3 Generalized Voronoi diagram

The concept of the ordinary planar Voronoi diagram can be extended to more complex set of generators. In particular, we can construct generalized Voronoi diagrams (GVD) for objects like lines, arcs, circles and polygons [12].

Let $L = \{L_1, L_2, L_3, \dots, L_n\} \subseteq \Re^2$ be a set where L_i can be a point, a line segment or an arc, such that they do not intersect each other, $L_i \cap L_j \neq 0$, for $i \neq j$. Let define the distance from a point p to L_i as the shortest distance between p and a point p_i on L_i by:

$$d_s(p, L_i) = \min_{\boldsymbol{x_i}} \|\boldsymbol{x} - \boldsymbol{x_i}\|; \boldsymbol{x_i} \in L_i, \tag{3}$$

where $\boldsymbol{x}$ and $\boldsymbol{x_i}$ are the position vectors of p and p_i, respectively. The Voronoi region $R(L_i)$ is given by:

$$R(L_i) = \{p; d_s(p, L_i) \leq d_s(p, L_j),\ j \neq i,\ j \in I_n\}. \tag{4}$$

The union set of all Voronoi regions given by $V(L) = \bigcup_{i=1}^{n} R(L_i)$ generates the line Voronoi diagram for the set L.

The Voronoi diagram structure for circles and polygons is easily obtained from the line Voronoi diagram just by considering the distance function from a point p to the nearest point p_i in the object. Figure 3 shows the GVD for a set of distinct generators.

It is important to mention here that the set of edges and vertices of a Voronoi diagram induces a graph structure [15], on which the techniques available for graphs may be applied.

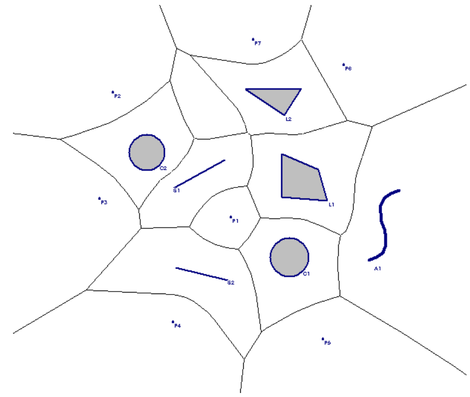

Figure 3: GVD for a set of generators.

2.4 Weighted Voronoi diagram

In the previous construction of Voronoi diagrams we have assumed *a priori* that all generators have the same status, or in different terms that all of them have the same weight. In certain cases it is appropriate to consider generators with distinct weights.

The underlying principle for weighted Voronoi diagrams (WVD) is similar to the OVD. Let us consider a set of non-collinear distinct points $P = \{p_1, p_2, \ldots, p_n\}$ in $\Re^2$ $(2 \leq n < \infty)$ and assign a weight to each point $p_i \in P$. The weight for each p_i will be represented by a parameter w_i.

The *weighted distance* from a point p to $p_i \in P$ is defined by

$$d_w(p, p_i) = \frac{1}{w_i} \|\boldsymbol{x} - \boldsymbol{x_i}\|, w_i > 0\,. \tag{5}$$

The weighted distance above is called *multiplicative weighted distance* (MW-distance) to distinguish it from other weighted distance definitions in the literature.

Based on the MW-distance the dominance regions of a point p_i over p_j can be defined. The dominance region of point p_i over p_j with weighted distance d_w is given by:

$$D_w(p_i, p_j) = \{p;\, d_w(p, p_i) \leq d_w(p, p_j),\, i \neq j\}\,. \tag{6}$$

The definition above can be extended to any weighted distance. In our particular case, we shall consider only the MD-distance, which becomes

$$D_w(p_i, p_j) = \{\boldsymbol{x};\, \frac{1}{w_i}\|\boldsymbol{x} - \boldsymbol{x_i}\| \leq \frac{1}{w_j}\|\boldsymbol{x} - \boldsymbol{x_j}\|,\, i \neq j\}\,. \tag{7}$$

The *weighted Voronoi region* for a generator p_i is determined by the intersection of dominance regions

$$R_w(p_i) = \bigcap_{j \in I_n \setminus \{i\}} D(p_i, p_j)\,. \tag{8}$$

The *weighted Voronoi diagram* (WVD) is obtained as

$$WVD(P) = \bigcup_i R_w(p_i)\,. \tag{9}$$

The multiplicative WVD need not be convex or connected and it may have holes [12]. Figure[2] 4 shows a multiplicative WVD for a set of point generators. It is important to note that the distance function, need not be a metric distance. This is where the strength of the Voronoi diagram structure resides.

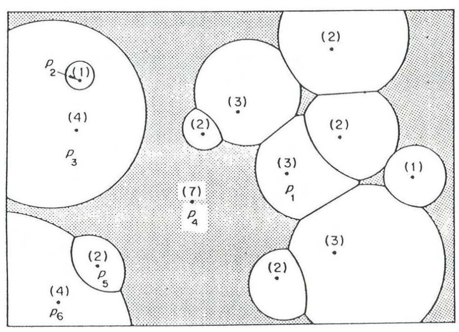

Figure 4: Multiplicative Weighted Voronoi diagram with weights in brackets.

3 Voronoi growth model of fire spread

Wildland fire is quite a complex problem, however here we will focus our attention to a single aspect that will serve as a subcomponent to a deeper and complete analysis concerning wildland fire modeling. Let us assume a homogeneous distribution of vegetation on a plane topography and without wind or other weather adversities. In such a case, the fire wavefront of a single fire ignition source is expected to spread out in a circular symmetry. If the spread rate is given by v for each fire source p_i, the wavefronts move radially with $r_i(t) = [vt\cos(s), vt\sin(s)]^T,\ s \in \mathcal{S}^1$.

When two fire ignition sources are lit simultaneously at time t_0, the fire wavefronts will propagate until $t_* > t_0$ where they initially touch each other.

[2]This figure is reproduced from [12] with permission of John Wiley & Sons Limited.

The point where the wavefronts meet is at the bisector of fire ignition sources. For $t > t_*$, the wavefronts intersect each other, but we assume that no fire can grow on an already burned out area. For $t > t_*$, the initial point of contact of the wave fronts splits and traces a line equidistant to the two fire sources, giving rise to a simple Voronoi Growth Model. For three non-collinear *appropriately placed* simultaneous fire ignition sources, the growth model gives rise to a *green island*, a region of vegetation that is no longer connected from the rest of the other vegetation areas due to the expanding fire.

Here we first describe the relative placement of three fire sources that are necessary to form a green island. Next, we extend the result for non point-like fire ignition sources and finally for multiple simultaneous fire ignition sources. Note that a green island itself will eventually be burned out if the fire is not suppressed. However, it is useful to determine their spatial and temporal formation as a green island can serve as *refuge zones* for wildlife.

Growth models produce spatial patterns as a consequence of the expansion of a set of points $P = \{p_1, p_2, \ldots, p_n\}$ in a region $S \subset \Re^2$. Let $B_i(t - \tau)$ be the burned vegetation at time t corresponding to the fire ignition source p_i that started at $t = \tau_i$. The growth model is said to be *uniform and isotropic* if the rate of expansion v is constant for all sources in all directions, i.e.,

$$B_i(t) = \begin{cases} \{p_i + [v(t' - \tau_i)\cos(s), v(t' - \tau_i)\sin(s)]^T : \\ \qquad\qquad \forall\, t' \le t, \text{ and } \; s \in \mathcal{S}^1\}, & \text{if } t > \tau, \\ \qquad\qquad\qquad p_i, & \text{if } t \le \tau. \end{cases}$$

Without loss of generality, let $v = 1$. The uniform growth model gives rise to a Voronoi diagram structure known as *Voronoi Growth Model* [12] or *cell model*, because the growth process in a particular direction ceases whenever it comes into contact with another growing region. When all sites are ignited at the same time, i.e., for all i and j, $\tau_i = \tau_j$, the growth model is said to be *simultaneous.* Again, without loss of generality, assume simultaneous growth models start at $t = 0$.

The burned vegetation set at time t is defined by:

$$B(t) = \bigcup_{i=1}^{n} B_i(t),$$

and the green vegetation set $G(t)$ at time t is given by:

$$G(t) = S \backslash B(t).$$

At $t = 0$, $G(t)$ is equal to S and is connected. With the Voronoi growth model, there exits a t_* such that the connectivity of $G(t)$ decreases from $t_*^- \le t_* \le t_*^+$. At t_*, the following is formed:

Definition 1 *A* **green island** I_g *is a disjoint subset of* $G(t)$.

Let $w_i(t) = \partial B_i(t)$ be the fire wavefront generated by the fire source p_i. Let c_{ij} be the point where two fire wavefronts, say $w_i(t_*)$ and $w_j(t_*)$ become tangent to each other at time t_*. The point c_{ij} lies on the bisector of p_i and p_j, i.e., $d(p_i, c_{ij}) = d(p_j, c_{ij})$, where $d(.,.)$ denotes the Euclidean distance.

For $t > t_*$, the point $c_{ij}(t)$ may give rise to two *fire vertices*, that will be denoted by $c^+_{ij}(t)$ and $c^-_{ij}(t)$. Note that $c^{\pm}_{ij}(t) = c^{\pm}_{ji}(t)$, and $d(c^{\pm}_{ij}(t), w_i) = d(c^{\pm}_{ij}(t), w_j) = 0$.

We will show below that three *appropriately spaced* ignition sources, p_i, p_j, and p_k, give rise to three inward fire vertices $c^-_{ij}(t)$, $c^-_{ik}(t)$, and $c^-_{jk}(t)$ that "pinch off" a region of the green vegetation and form a green island. Eventually, there exists a t^* such that $c^-_{ij}(t)$, $c^-_{ik}(t)$, and $c^-_{jk}(t)$ converge onto q_{ijk}, a Voronoi vertex, as t goes to t^*. At this time, the green island has been fully consumed by the fire.

Note that $d(c^-_{ij}, q_{ijk})$ decreases as t goes to t^*, and that $c^-_{ij} \neq \emptyset$ for $t \in (t_*, t^*)$. At t^*, the green island associated with q_{ijk} is burned up.

4 Green Island Formation

In this section, we consider the formation of green islands when a simultaneous uniform Voronoi growth model is in effect. At the end of this section, we relax the simultaneity assumption.

4.1 Simple green island

I_g is said to be *simple* when it has been formed out of three fire sources. Otherwise, it is said to be *complex*. A complex green island can breakdown to form other complex or simple islands.

We may view $B_i(t)$ as obstacles (that grew from the point sites) and consider the Generalized Voronoi Diagram (GVD) [4, 17] in S with obstacles $B_i(t)$. For the uniform simultaneous growth model, the resulting GVD will be a subset of the conventional Voronoi diagram. Since the GVD is connected in each connected piece of free space, each green island I_g has its own unique GVD structure. There is a duality between the generalized Voronoi diagram (GVD) of the burned regions and the connectivity of the free space, as indicated in the following:

Proposition 1 *Given three fire ignition sources* $\{p_i, p_j, p_k\}$, *a simple green island* I_g *is formed if and only if the only fire vertices connected to* q_{ijk} *are* $\{c^-_{ij}, c^-_{jk}, c^-_{ik}\}$.

Proof Consider the GVD where $B_h(t)$ for all h are the obstacles. Recall that the GVD is connected for each connected region of free space [4]. Since a green island has been formed, it has its own connected GVD. Since this island was formed by three obstacles $B_i(t), B_j(t), B_k(t)$, there exists a Voronoi vertex q_{ijk} that is equidistant to $\{p_i, p_j, p_k\}$ and cannot be adjacent to any other Voronoi vertex. Therefore, the only adjacent nodes to q_{ijk} are boundary nodes, nodes where the distance to $B_i(t), B_j(t)$ (and all other pair-wise combinations) is zero. In other words, the only adjacent nodes to q_{ijk} are c_{ij}^-, c_{jk}^-, and c_{ki}^-.

Likewise, if $\{c_{ij}^-, c_{jk}^-, c_{ki}^-\}$ are the only connected nodes to q_{ijk}, since the distance to the obstacles $B_i(t), B_j(t), B_k(t)$ for each $\{c_{ij}^-, c_{jk}^-, c_{ki}^-\}$ is zero, q_{ijk}, $\{c_{ij}^-, c_{jk}^-, c_{ki}^-\}$ form a disconnected GVD structure and thus a green island has formed. □

Now, we consider the appropriate placement for $\{p_i, p_j, p_k\}$ to allow the formation of a green island.

Theorem 1 *For simultaneous uniform growth, a green island is formed by $\{p_i, p_j, p_k\}$ if and only if q_{ijk} lies in the convex hull of $\{p_i, p_j, p_k\}$.*

Proof First, consider the converse – a green island is formed. By Proposition 1, we know that a triangle formed by $\{c_{ij}, c_{jk}, c_{ki}\}$ will exist because there is a green island. Also, note that the triangle formed by $\{p_i, p_j, p_k\}$ is similar to the triangle formed by $\{c_{ij}, c_{jk}, c_{ki}\}$. See Figure 5.

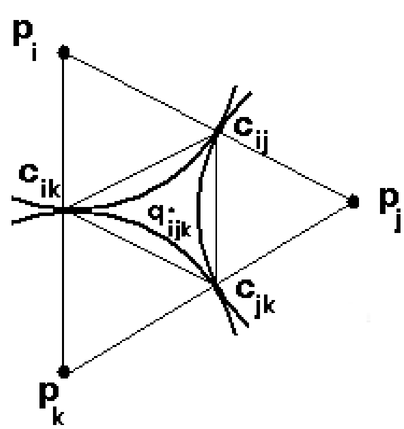

Figure 5: Green island formation.

We will now show that the triangle formed by $\{c_{ij}, c_{jk}, c_{ki}\}$ must contain the Voronoi vertex q_{ijk}. Actually, we are going to show that the hyperbolic triangle inside of triangle $\{c_{ij}, c_{jk}, c_{ki}\}$ contains q_{ijk}. The three curved edges that form the hyperbolic triangle comes from $w_i(t), w_j(t)$, and $w_k(t)$.

Consider the functions

$$\begin{aligned} G_{ij}(x) &= (d_i - d_j)(x), \\ G_{ik}(x) &= (d_i - d_k)(x), \\ G_{jk}(x) &= (d_j - d_k)(x). \end{aligned}$$

Now let us define the function

$$G(x) = \begin{cases} G_{ij}^2(x) + G_{jk}^2(x) & \text{if } d_j(x) \le d_i(x) \text{ and } d_j(x) < d_k(x) \\ G_{ij}^2(x) + G_{ik}^2(x) & \text{if } d_i(x) \le d_j(x) \text{ and } d_i(x) < d_k(x) \\ G_{jk}^2(x) + G_{ik}^2(x) & \text{if } d_k(x) \le d_i(x) \text{ and } d_k(x) < d_j(x) \end{cases}$$

So, on the $w_i(t)$ wavefront, $G(x) = G_{ij}^2(x) + G_{ik}^2(x)$. For all $x \in w_i(t)$, and

$$\nabla G(x) = 2(d_i(x) - d_j(x))(\nabla d_i(x) - \nabla d_j(x)) + 2(d_i(x) - d_k(x))(\nabla d_i(x) - \nabla d_k(x)),$$

taking the inner product of $\nabla G(x)$ and $\nabla d_i(x)$, for all points x on w_i, we get

$$\begin{aligned} \nabla G(x) \cdot \nabla d_i(x) &= \langle d_i(x), 2(d_i(x) - d_j(x))(\nabla d_i(x) - \nabla d_j(x)) \\ &\quad + 2(d_i(x) - d_k(x))(\nabla d_i(x) - \nabla d_k(x))\rangle \\ &= 2(d_i(x) - d_j(x))(1 - \langle \nabla d_i(x), d_j(x)\rangle) \\ &\quad + 2(d_i(x) - d_k(x))(1 - \langle \nabla d_i(x), d_k(x)\rangle) \end{aligned}$$

Since $x \in w_i(t)$, $d_i(x) - d_j(x) < 0$, and $d_i(x) - d_k(x) < 0$, and, furthermore, by definition $-1 < \langle \nabla d_i(x), d_j(x)\rangle < 1$ and $-1 < \langle \nabla d_i(x), d_k(x)\rangle < 1$, the above expression for $\langle \nabla d_i(x), \nabla G(x)\rangle$ must be negative.

In fact, $\langle \nabla d_j(x), \nabla G(x)\rangle$ and $\langle \nabla d_k(x), \nabla G(x)\rangle$ are also negative. Therefore, G decreases from the boundary to the interior of the triangle. This is true for all $t < t^*$ and thus $G(x)$ monotonically decreases for all paths from the boundary towards the interior of the triangle. Since $G(x)$ is a continuous function that is decreasing for all paths directed into the triangle, it must have a unique minimum. Furthermore, G is bounded below by zero, so the minimum value of G is zero. Finally, since G vanishes when $x = q_{ijk}$ (and this minimum is unique), q_{ijk} must be contained inside of the hyperbolic triangle formed by $w_i(t), w_j(t)$, and $w_k(t)$ whose vertices are $\{c_{ij}, c_{jk}, c_{ki}\}$. This triangle is in the convex hull of $\{c_{ij}, c_{jk}, c_{ki}\}$ which is contained in the convex hull of $\{p_i, p_j, p_k\}$. Alternatively, when the gradient from each fire source positively span the convex hull[4] of $\{p_i, p_j, p_k\}$ a simple green island will be formed.

Now we consider the forward direction: suppose q_{ijk} is contained in the convex hull of $\{p_i, p_j, p_k\}$. Consider the triangle formed by the fire sources. As q_{ijk} is inside this triangle, it is possible to find along the Voronoi edges the intersecting points of the edges with each side of the triangle $\{p_i, p_j, p_k\}$. By Voronoi growth construction these points correspond to c_{ij}, c_{ik} and c_{jk}, respectively. As q_{ijk} is inside the triangle formed by c_{ij}, c_{ik}, c_{jk}, then each one of the c_{ij}'s will give rise to an inward fire vertex. Therefore, by Proposition 1 a green island will be formed. □.

It is worth noting that the above result indicates that three collinear fire ignitions sources cannot give rise to a green island, even when one assumes there is a Voronoi vertex at infinity.

4.2 Non-simultaneous and Non-point-like fire ignition sources

The above result can be extended to non-simultaneous and to different shapes of the fire ignition sources. In the case of non-simultaneity of point-like fire ignition sources, the resulting Voronoi diagram is *not* a subset of the conventional Voronoi diagram. Furthermore, the location of the resulting edges of the generalized Voronoi diagram is a function of time t.

Corollary 1 *At a* $t = \max\{\tau_i, \tau_j, \tau_k\}$*, a green island is formed by* $\{p_i, p_j, p_k\}$ *if and only if* $q_{ijk}(t)$ *lies in the convex hull of* $\{p_i, p_j, p_k\}$*.*

Proof The proof of this Corollary follows directly from application of Theorem 1. □

An extension of the previous results can be applied for the case when the fire ignition sources are no longer point-like only. In other words, in some situations, as in prescribed fires, the ignition may start as a line, arc, circle or even in a polygonal shape. If we consider every point on the object as a source of fire, the fire wavefront will spread out according to Huygens' Principle [5].

The fire ignition sources will generate a generalized Voronoi diagram (GVD) and the GVD remains the same as the fire grows. A green island will be formed when part of the GVD looses its connectivity in a similar fashion as in the point-like fire source situation.

The Voronoi edges are the growth limit between the fire wavefronts. Once the wavefront reaches an edge a fire vertex is formed. As such, a green island can be formed in much the same way as for point-like fire ignition sources.

At each point on the fire wavefront the propagation direction is pointing along the gradient. Let l_i, l_j, l_k be three fire ignition sources (point, line, arc, etc.) and let q_{ijk} be the GVD vertex formed by them.

Corollary 2 *For simultaneous uniform fire growth, a green island is formed by l_i, l_j, l_k if and only if q_{ijk} lies inside the convex hull of the gradients for the three fire wavefronts at q_{ijk}.*

Proof Essentially, the proof of this Corollary follows from Theorem 1. Consider a Voronoi vertex q_{ijk} associated with three or more convex obstacles. Without loss of generality, let p_i, p_j and p_k be the three closest points on the three defining obstacles. By the property of convex sets and distance function, q_{ijk} must lie in the convex hull of p_i, p_j and p_k. By themselves, p_i, p_j and p_k would from a green island. Furthermore, the wavefront of the obstacles associated with these points would reach the Voronoi vertex first, so p_i, p_j and p_k indeed pinches off a region of free space forming a green island. □

4.3 Complex green island

Complex green islands can be formed at time t for a set of simultaneous fire sources even when no three adjacent fire sources do form an island by themselves. In this subsection we provide a constructive algorithm to find out when a green island is formed at time t for a set of simultaneous fire sources.

Let $P = \{p_1, p_2, \dots, p_n\}$ be a set of n simultaneous fire ignition sources. Let $t = t^*$ be such that the growth radius $r(t^*) = R$.

Determine all sets of adjacent fire sources such that

$$\frac{d(p_i, p_j)}{2} \leq R, \forall i, j;\ i \neq j. \tag{10}$$

The fire sources that satisfy the above condition have their wavefronts such that $w_i \bigcap w_j \neq \emptyset$. Let us suppose that $P^* = \{P_1, P_2, , \dots, P_m\}, m \leq n$, is the set of all disjoint sets P_i composed of a sequence of adjacent fire sources that fulfill Equation 10. For each set $P_i \in P^*$, search the cycles with at least three fire sources. Let the set of all possible cycles of P_i be $C = \{C_1, C_2, \dots, C_k\}$. Construct a simple polygon associated with each cycle $C_l \in C$, where the fire sources are the polygon's vertices. Let define an *fire empty polygon* as a simple polygon without any fire source in its interior.

Theorem 2 *A complex green island is formed at time t^* if and only if there is at least one Voronoi vertex, say q_{ijk}, in the interior of a fire empty polygon formed from a cycle $C_l \in C$, such that*

$$d(p_i, q_{ijk}) > R,\ p_i \in C_l.$$

Proof Let $C_l = \{p_1, p_2, \ldots, p_s\}$, where $s \leq n$, be a cycle of adjacent fire sources at time t^* satisfying equation 10 and forming a fire empty polygon. At t^* all adjacent fire sources in C_l are such that $w_i \bigcap w_j \neq \emptyset$. Thus $d(p_i, p_j) \leq 2R$. Suppose there is a Voronoi vertex q_{ijk} in the interior of the fire empty polygon formed by the sequence $p_1 p_2 \ldots p_s$. If no fire wavefront has reached the Voronoi vertex q_{ijk} at t^*, than it satisfies the condition $d(p_i, q_{ijk}) > R$ for all $p_i \in C_l$. However, as the Voronoi vertices are the last points to be burned, a complex green island is formed around q_{ijk} at t^*, or equivalently, when $r(t^*) = R$. Now, let us assume that an island exist around a Voronoi vertex q_{ijk}. That means no fire wavefront has reached q_{ijk} yet and so $d(p_i, q_{ijk}) > R$. As an island is formed, there is no connectivity between the Voronoi edges inside the island and outside it. Hence there should be at least one cycle C_l such that it encompass the Voronoi vertex q_{ijk}. If C_l is not empty there will be a sub-cycle $C_e \subset C_l$ such that C_e is an empty cycle. $\square$

Unfortunately, for complex green island the fact that a Voronoi vertex does not satisfy Theorem 1 is not sufficient to guarantee that it does not belong to a green island. Therefore, we can not directly apply Theorem 1 for complex green island. Nevertheless, it is easy to see that Theorem 2 reduces to Theorem 1 when there are three ignition sources only.

Figure 6 shows a complex green island where some Voronoi vertices are connected to each other only. Ultimately, all Voronoi vertices within a complex green island are connected to inward fire vertices as terminal nodes. Figure 7 shows two fire empty polygons constructed from cycles in complex green islands.

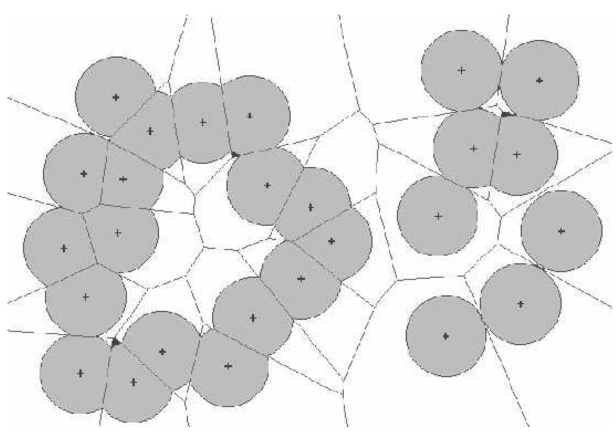

Figure 6: Complex green islands for a set of simultaneous fire sources.

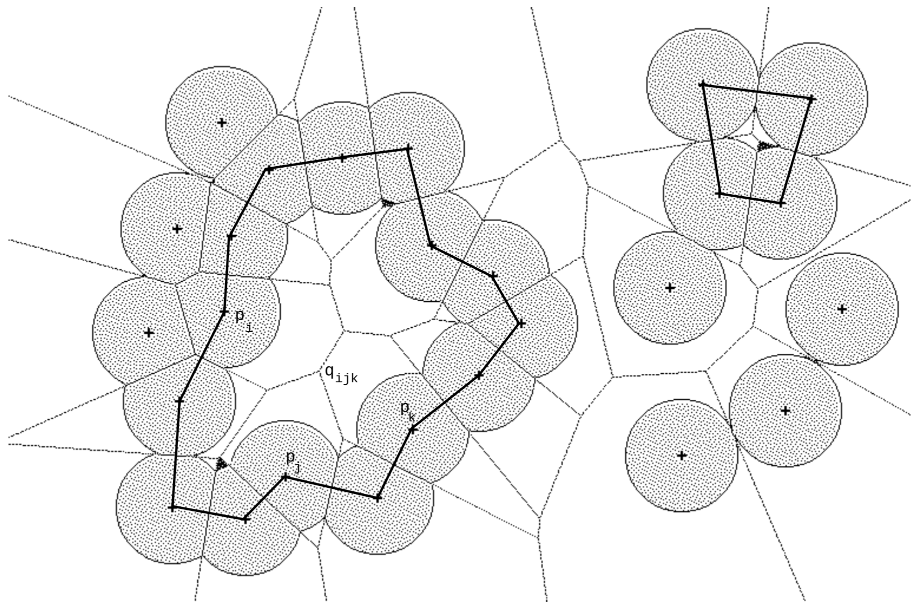

Figure 7: Empty polygons for the complex green islands.

5 Comments and Conclusions

In this paper the Voronoi Growth technique has been applied to model the geometry of burned areas due to surface fire. In particular, we have discussed the specific geometrical problem of simple and complex green island formations for simultaneous and non-simultaneous fire ignition sources. The spatial and temporal identification of a green island is an important aspect for simulation of prescribed fires and real-time fire spread monitoring [5], because the green islands form the *refuge zones* for the wildlife. Here Theorem 1 allows to predict when a green island will be formed from three fire ignition sources and Theorem 2 provides the means to identify the formation of complex green islands at a time t.

In a more realistic situation where the fire ignition sources are non-simultaneous with distinct ignition patterns (lines, arcs, polygons), in the presence of wind and a non-planar topography the Voronoi Growth still shows a rich structure to further model the landscape geometry of wildland fires [17]. For instance, in the presence of wind, the landscape geometry can be modeled using the approach of *Voronoi diagrams in the river* [20] and for terrain with slopes, the recent work on *skew Voronoi diagrams* [1] seem an interesting ones.

Even when these strong assumptions are present, the wildland fire still exhibit similar features as the ones predicted in our approach. For example, the Figure 8 shows the 1988 Yellowstone National Park fire resemble the Voronoi growth model and green island formation.

Figure 8: 1988 Yellowstone National Park Fire (Photo by R. Hartford).

The fire ignition sources can have different shapes, e.g., lines, arcs, circles or polygons. These patterns are more often used in prescribed fires in national parks, in farms to clean the field or as process to ease the cropping in large sugar plantation in Brazil. The Voronoi Growth Model defined in Section 3 can be extended to a Generalized Voronoi Growth Model, where the Voronoi generators can be any set of these objects. Although the fire sources may actually be a combination of these patterns, most of the proposed wildland fire models in the literature have considered only point-like fire sources.

Another feature of modeling wildland fires with Voronoi diagrams is that it forms a graph structure, which makes simple to determine the shortest path with maximal clearance between two points among the fire wavefronts. This path corresponds to the safest trajectory for firefighters to move during rescue operations and planning strategies to fire control and suppression.

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